# FUNCTIONAL INEQUALITIES IN THE ABSENCE OF CONVEXITY AND LOWER SEMICONTINUITY WITH APPLICATIONS TO OPTIMIZATION

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Abstract. In this paper we extend some results in [5] to the setting of functional inequalities when the standard assumptions of convexity and lower semicontinuity of the involved mappings are absent. This extension is achieved under certain condition relative to the second conjugate of the involved functions. The main result of this paper, Theorem 1, is applied to derive some subdifferential calculus rules, different generalizations of Farkas lemma for nonconvex systems, as well as some optimality conditions and duality theory for infinite nonconvex optimization problems. Several examples are given to illustrate the significance of the main results and also to point out the potential of their applications to get various extensions of Farkas-type results and to the study of other classes of problems such as variational inequalities and equilibrium models.

## 1. Introduction

Given two convex lower semicontinuous extended real-valued functions $F$ and $h$, defined on locally convex spaces, we provided in [5] a dual transcription of the functional inequality

$$(*) \qquad F(0,\cdot) \geq h(\cdot),$$

in terms of the Legendre-Fenchel conjugates of $F$ and $h$, and applied this result to convex subdifferential calculus, subgradients-based optimality conditions, Farkas-type results, and, in the optimization field, to linear, convex, semi-definite, and DC problems. The main feature of our approach there was the absence of the so-called topological constraint qualifications and closedness conditions in the hypotheses.

In many situations the well-known constraint qualifications (CQ), as generalized Slater-type/interior-type, Mangasarian-Fromovitz CQ, Robinson-type CQ, or Attouch-Brezis CQ, fail to hold. This is the case in many classes of scalarized forms of (convex) vector optimization problems, in semi-definite programs, bilevel programming problems (see, e.g., [3], [6], [34], etc.). Because of that, in the last decades many efforts were devoted to establish mathematical tools for such classes of problems (e.g., [1], [2], [5], [6], [8], [20], [22], [25], [29], [30], [33], etc.).

Nowadays, in science and technology there is a huge number of practical problems that can be modelled as nonconvex optimization problems (see [14], [15], [24], and references therein).

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In the present paper, we go a step further than what is done in [5] by relaxing the convexity and the lower semicontinuity on the function $F$ in the left hand side of $(*)$. Doing so, we use convex tools for nonconvex problems: a tendency whose importance increases nowadays. Even more, we characterize in Theorem 1 the class of functions $F$ for which the dual transcription of $(*)$ obtained in [5] does work. We show that the class of such functions $F$ goes far beyond the usual one of convex and lower semicontinuous extended real valued mappings. In fact, this extension is achieved under certain condition relative to the second Legendre-Fenchel conjugates of the mappings $F$ and $F(0,\cdot)$. A dual geometrical description of this property is given in Proposition 3.

As consequences of Theorem 1, we obtain extensions of the basic convex subdifferential calculus formulas for non necessarily convex functions (Theorem 2 and Proposition 2), Farkas-type results for nonconvex systems (Propositions 4 and 5), optimality conditions for non-convex optimization problems (Propositions 6, 7, 9, and 10), from which we derive the corresponding recent basic results in the convex setting (Corollaries 1 and 2).

In the same way, we provide duality theorems for nonconvex optimization problems (Proposition 8, Corollary 3) that cover some recent results in the convex case (Corollary 4).

The results presented in this paper are new, up the knowledge of the authors, and they extend in different directions some relevant results in the literature, as [4], [9]-[13], [16]-[22]. The extensions we propose here are such that typical assumptions as the convexity and/or lower semicontinuity of the involved functions, the closedness-type constraint qualifications conditions are absent. Besides this, Examples 1-3, in Section 3, also show the potential of Theorem 1 to get further generalizations of Farkas-type theorems and of other results in the field of variational inequalities and equilibrium problems, always in the absence of convexity, lower semicontinuity and of any closedness/qualification conditions.

## 2. Notation and preliminary results

Let $X$ be a locally convex Hausdorff topological vector space (l.c.H.t.v.s.) whose topological dual is denoted by $X^*$. The only topology we consider on $X^*$ is the $w^*$-topology.

Given two nonempty sets $A$ and $B$ in $X$ (or in $X^*$), we define the *algebraic sum* by

$$A + B := \{a + b \mid a \in A,\ b \in B\}, \quad A + \emptyset := \emptyset + A := \emptyset, \tag{2.1}$$

and we set $x + A := \{x\} + A$.

Through the paper we adopt the rule $(+\infty) - (+\infty) = +\infty$.

We denote by $\operatorname{co} A$, $\operatorname{cone} A$ and $\operatorname{cl} A$, or indistinctly by $\overline{A}$, the convex hull, the conical convex hull and the closure of $A$, respectively.

Given a function $h \in (\mathbb{R} \cup \{+\infty\})^X$, its *(effective) domain*, *epigraph*, and *level set* are respectively defined by

$$\begin{aligned}
\operatorname{dom} h &:= \{x \in X : \ h(x) < +\infty\},\\
\operatorname{epi} h &:= \{(x,\alpha) \in X \times \mathbb{R} : \ h(x) \le \alpha\},\\
[h \le \alpha] &:= \{x \in X : \ h(x) \le \alpha\}.
\end{aligned}$$

The function $h \in (\mathbb{R} \cup \{+\infty\})^X$ is *proper* if $\operatorname{dom} h \neq \emptyset$, it is *convex* if $\operatorname{epi} h$ is convex, and it is *lower semicontinuous* (lsc, in brief) if $\operatorname{epi} h$ is closed.

The *lower semicontinuous envelope* of $h$ is the function $\overline{h} \in (\mathbb{R} \cup \{\pm\infty\})^X$ defined by

$$\overline{h}(x) := \inf\{t : \ (x,t) \in \operatorname{cl}(\operatorname{epi} h)\}.$$

Clearly we have $\operatorname{epi} \overline{h} = \overline{\operatorname{epi} h}$, which implies that $\overline{h}$ is the greatest lsc function minorizing $h$; so $\overline{h} \leq h$. If $h$ is convex, then $\overline{h}$ is also convex, and then $\overline{h}$ does not take the value $-\infty$ if and only if $h$ admits a continuous affine minorant.

Given $h \in (\mathbb{R} \cup \{+\infty\})^X$, the *lsc convex hull* of $h$ is the convex lsc function $\overline{\operatorname{co}} h \in (\mathbb{R} \cup \{\pm\infty\})^X$ such that

$$\operatorname{epi}(\overline{\operatorname{co}} h) = \overline{\operatorname{co}}(\operatorname{epi} h).$$

Obviously $\overline{\operatorname{co}} h \leq \overline{h} \leq h$.

We shall denote by $\Gamma(X)$ the class of all the proper lsc convex functions on $X$. The set $\Gamma(X^*)$ is defined similarly.

Given $h \in (\mathbb{R} \cup \{+\infty\})^X$, the *Legendre-Fenchel conjugate* of $h$ is the function $h^* \in (\mathbb{R} \cup \{\pm\infty\})^{X^*}$ given by

$$h^*(x^*) = \sup\{\langle x^*, x\rangle - h(x) : \ x \in X\}.$$

The function $h^*$ is convex and lsc. If $\operatorname{dom} h = \emptyset$ we have $h^* = \{-\infty\}^X$ (i.e., $h^*(x^*) = -\infty$ for all $x^* \in X^*$). Moreover, $h^* \in \Gamma(X^*)$ if and only if $\operatorname{dom} h \neq \emptyset$ and $h$ admits a continuous affine minorant.

The *biconjugate* of $h$ is the function $h^{**} \in (\mathbb{R} \cup \{\pm\infty\})^X$ given by

$$h^{**}(x) := \sup\{\langle x^*, y\rangle - h^*(x^*) : \ x^* \in X^*\}.$$

We have

$$\{h \in (\mathbb{R} \cup \{+\infty\})^X : \ h = h^{**}\} = \Gamma(X) \cup \{+\infty\}^X.$$

Moreover, $h^{**} \leq \overline{\operatorname{co}} h$, and the equality holds if $h$ admits a continuous affine minorant.

The *indicator* function of $A \subset X$ is defined as

$$i_A(x) := \begin{cases} 0, & \text{if } x \in A, \\ +\infty, & \text{if } x \in X \setminus A. \end{cases}$$

If $A \neq \emptyset$ the conjugate of $i_A$ is the *support function* of $A$, $i_A^* : X^* \to \mathbb{R} \cup \{+\infty\}$.

Given $a \in h^{-1}(\mathbb{R})$ and $\varepsilon \geq 0$, the $\varepsilon$-*subdifferential* of $h$ at the point $a$ is defined by

$$\partial_\varepsilon h(a) = \{x^* \in X^* : h(x) - h(a) \geq \langle x^*, x - a\rangle - \varepsilon, \forall x \in X\}.$$

One has

$$\partial_\varepsilon h(a) = [h^* - \langle \cdot, a\rangle \leq \varepsilon - h(a)] = \{x^* \in X^* : h^*(x^*) - \langle x^*, a\rangle \leq \varepsilon - h(a)\}.$$

If $a \notin h^{-1}(\mathbb{R})$, set $\partial_\varepsilon h(a) = \emptyset$. If $h \in (\mathbb{R} \cup \{+\infty\})^X$ is convex, then we have $\partial_\varepsilon h(x) \neq \emptyset$ for all $\varepsilon > 0$ if and only if $h$ is lsc at $x$.

The $\varepsilon$-*normal set* to a nonempty set $A$ at a point $a \in A$ is defined by

$$\mathrm{N}_\varepsilon(A, a) = \partial_\varepsilon i_A(a).$$

The *Young-Fenchel inequality*

$$f^*(x^*) \geq \langle x^*, a\rangle - f(a)$$

always holds. The equality holds if and only if $x^* \in \partial f(a) := \partial_0 f(a)$.

The *limit superior* when $\eta \to 0_+$ of the family $(A_\eta)_{\eta>0}$ of subsets of a topological space is defined (in terms of generalized sequences or nets) by

$$\limsup_{\eta\to 0_+} A_\eta := \left\{ \lim_{i\in I} a_i : a_i \in A_{\eta_i}, \forall i \in I, \text{ and } \eta_i \to 0_+ \right\},$$

where $\eta_i \to 0_+$ means that $(\eta_i)_{i\in I} \to 0$ and $\eta_i > 0$, $\forall i \in I$.

Let $U$ be another l.c.H.t.v.s. whose topological dual is denoted by $U^*$, and let us consider $F \in \Gamma(U \times X)$. In [5] we established the following result:

**Proposition 1.** *Let* $F \in \Gamma(U \times X)$ *with* $\{x \in X : F(0,x) < +\infty\} \neq \emptyset$. *For any* $h \in \Gamma(X)$, *the following statements are equivalent*:
(a) $F(0,x) \geq h(x)$, *for all* $x \in X$.
(b) *For every* $x^* \in \operatorname{dom} h^*$, *there exists a net* $(u_i^*, x_i^*, \varepsilon_i)_{i\in I} \subset U^* \times X^* \times \mathbb{R}$ *such that*

$$F^*(u_i^*, x_i^*) \leq h^*(x^*) + \varepsilon_i, \text{ for all } i \in I,$$

*and*

$$(x_i^*, \varepsilon_i) \to (x^*, 0_+).$$

## 3. Functional inequalities involving non necessarily convex neither lsc mappings

The following theorem constitutes an extension of Proposition 1 to a function $F$ which is neither convex nor lower semicontinuous, but under certain specific requirement to be satisfied by the second conjugate $F^{**}$. In fact, it delivers a characterization of that requirement.

**Theorem 1.** *Let* $F : U \times X \to \mathbb{R}\cup\{+\infty\}$ *such that* $F(0,\cdot)$ *is proper and* $\operatorname{dom} F^* \neq \emptyset$. *Then, the following statements are equivalent*:
(a) $F^{**}(0,\cdot) = (F(0,\cdot))^{**}$.
(b) *For any* $h \in \Gamma(X)$,

$$F(0,x) \geq h(x),\ \forall x \in X \iff \left\{ \begin{array}{l} \forall x^* \in \operatorname{dom} h^*, \textit{ there exists a net} \\ (u_i^*, x_i^*, \varepsilon_i)_{i\in I} \subset U^* \times X^* \times \mathbb{R} \textit{ such that} \\ F^*(u_i^*, x_i^*) \leq h^*(x^*) + \varepsilon_i,\ \forall i \in I,\ \textit{and} \\ \lim_{i\in I}(x_i^*, \varepsilon_i) = (x^*, 0_+). \end{array} \right\}$$

*Proof.* Assume that (a) holds and let $h \in \Gamma(X)$ satisfying $F(0,\cdot) \geq h$. Taking biconjugates in both sides we get $(F(0,\cdot))^{**} \geq h^{**} = h$, and by (a), $F^{**}(0,\cdot) \geq h$. Applying Proposition 1 with $F^{**} \in \Gamma(U \times X)$ playing the role of $F$ (observe that $\{x \in X : F^{**}(0,x) < +\infty\} \subset \operatorname{dom} F(0,\cdot) \neq \emptyset$), and recalling that $F^{***} = F^*$, we get the implication [$\Rightarrow$] in (b).

Assume now that, for a given $h \in \Gamma(X)$, the right hand side in the equivalence (b) holds. Again, by Proposition 1 applied to $F^{**}$ we get

$$F(0,x) \geq F^{**}(0,x) \geq h(x), \quad \forall x \in X.$$

Thus, we have that the converse implication [$\Leftarrow$] in (b) also holds.

Assume now that (b) holds. For any $(x^*, r) \in X^* \times \mathbb{R}$ such that

$$F(0,\cdot) \geq \langle x^*, \cdot\rangle - r, \tag{3.1}$$

let us apply (b) with $h = \langle x^*, \cdot\rangle - r$ to conclude the existence of a net $(u_i^*, x_i^*, \varepsilon_i)_{i\in I} \subset U^* \times X^* \times \mathbb{R}$ such that

$$F^*(u_i^*, x_i^*) \leq h^*(x^*) + \varepsilon_i = r + \varepsilon_i,\ \forall i \in I,$$

and

$$\lim_{i\in I}\,(x_i^*,\varepsilon_i)=(x^*,0_+).$$

Thus we have, for any $x\in X$,

$$F^{**}(0,x)\geq\langle x_i^*,x\rangle-F^*(u_i^*,x_i^*)\geq\langle x_i^*,x\rangle-r-\varepsilon_i,\ \forall i\in I,$$

and, passing to the limit on $i\in I$,

$$F^{**}(0,\cdot)\geq\langle x^*,\cdot\rangle-r. \tag{3.2}$$

Since (3.2) holds whenever $(x^*,r)$ satisfies (3.1), we get

$$\begin{aligned}F^{**}(0,\cdot)\ &\geq\ \sup\{\langle x^*,\cdot\rangle-r:\ (x^*,r)\ \text{satisfies (3.1)}\}\\ &=\ (F(0,\cdot))^{**}.\end{aligned}$$

Since $F^{**}(0,\cdot)$ is convex, lsc, and $F^{**}(0,\cdot)\leq F(0,\cdot)$, one has always $F^{**}(0,\cdot)\leq F(0,\cdot)^{**}$ and, finally, (a) holds. □

As the following examples illustrate, one easily realizes that the class of mappings $F$ satisfying condition (a) of Theorem 1 goes far beyond $\Gamma(U\times X)$.

**Example 1.** Given a proper function $f:U\to\mathbb{R}\cup\{+\infty\}$ such that $f^*$ is proper, and a linear continuous map $A:X\to U$, whose adjoint operator is denoted by $A^*$, let us consider

$$F(u,x):=f(u+Ax),\qquad (u,x)\in U\times X.$$

We thus have

$$F^*(u^*,x^*)=\begin{cases}f^*(u^*), & \text{if } A^*u^*=x^*,\\ +\infty, & \text{otherwise,}\end{cases}\qquad (u^*,x^*)\in U^*\times X^*,$$

and

$$F^{**}(u,x)=f^{**}(u+Ax),\qquad (u,x)\in U\times X.$$

Assuming that $F(0,\cdot)\equiv f\circ A$ is proper, that $(\operatorname{dom} f^*)\cap(\operatorname{dom} A^*)\neq\emptyset$, and that

$$(F(0,\cdot))^{**}\equiv(f\circ A)^{**}=f^{**}\circ A\equiv F^{**}(0,\cdot),$$

we are in position to apply Theorem 1 with $f$ possibly nonconvex. In such a way we get that for any $h\in\Gamma(X)$,

$$f\circ A\geq h\iff\left\{\begin{array}{l}\forall x^*\in\operatorname{dom} h^*,\ \text{there exists a net}\\ (u_i^*,\varepsilon_i)_{i\in I}\subset U^*\times\mathbb{R}\ \text{such that}\\ f^*(u_i^*)\leq h^*(x^*)+\varepsilon_i,\ \forall i\in I,\\ \text{and }\lim_{i\in I}(A^*u_i^*,\varepsilon_i)=(x^*,0_+).\end{array}\right\}$$

The case when $A$ is an homeomorphism (regular) is of particular interest as the relation $(f\circ A)^{**}=f^{**}\circ A$ holds for any function $f:U\to\mathbb{R}\cup\{+\infty\}$. This is the case when $U=X$ and $A$ is the identity map.

**Example 2.** Given two proper functions $f:X\to\mathbb{R}\cup\{+\infty\}$, $g:U\to\mathbb{R}\cup\{+\infty\}$, such that $f^*$ is proper and $g(0)=g^{**}(0)=0$, let us set

$$F(u,x)=f(x)+g(u),\quad (u,x)\in U\times X.$$

One has

$$F^*(u^*,x^*)=f^*(x^*)+g^*(u^*),\ (u^*,x^*)\in U^*\times X^*$$

and

$$F^{**}(u,x)=f^{**}(x)+g^{**}(u),\ (u,x)\in U\times X,$$

and so,

$$(F(0,.))^{**} = f^{**}(\cdot) + g^{**}(0) = F^{**}(0,.).$$

Since $f^*$ *is assume to be proper and* $g^{**}(0) \in \mathbb{R}$, we have that $F^*$ is proper. It then follows from Theorem 1 that, for any $h \in \Gamma(X)$,

$$f \geq h \iff \left\{ \begin{array}{l} \forall x^* \in \operatorname{dom} h^*, \text{ there exists a net} \\ (u_i^*, x_i^*, \varepsilon_i)_{i\in I} \subset U^* \times X^* \times \mathbb{R} \text{ such that} \\ f^*(x_i^*) + g^*(u_i^*) \leq h^*(x^*) + \varepsilon_i, \ \forall i \in I, \\ \text{and } \lim_{i\in I}(x_i^*, \varepsilon_i) = (x^*, 0_+). \end{array} \right\}$$

Observe that for $g \equiv 0$ we get

$$f \geq h \iff \left\{ \begin{array}{l} \forall x^* \in \operatorname{dom} h^*, \text{ there exists a net} \\ (x_i^*, \varepsilon_i)_{i\in I} \subset X^* \times \mathbb{R} \text{ such that} \\ f^*(x_i^*) \leq h^*(x^*) + \varepsilon_i, \ \forall i \in I, \\ \text{and } \lim_{i\in I}(x_i^*, \varepsilon_i) = (x^*, 0_+). \end{array} \right\}$$

The equivalence just above is in fact a consequence of $h \in \Gamma(X)$ and that $f^*$ is lsc on $X^*$.

**Example 3.** Given $f : X \times X \to \mathbb{R} \cup \{+\infty\}$, $a : X \to \mathbb{R} \cup \{+\infty\}$, $b \in \Gamma(X)$, and $K \subset X$, let us consider the problem

$(P)$ Find $\overline{x} \in K \cap \operatorname{dom} a \cap \operatorname{dom} b$ such that $f(\overline{x}, x) + a(x) \geq b(x) + a(\overline{x}) - b(\overline{x}), \ \forall x \in K.$

Problem $(P)$ extends many generalized equilibrium problems. This is, for instance, the case in relation to problem (GEP) in [7].

In order to formulate a dual expression for $(P)$ via Theorem 1, we introduce the following perturbation function associated with $\overline{x} \in K$

$$F(u, x) := f_{\overline{x}}(x) + (a + i_K)(u + x), \qquad (u, x) \in X \times X,$$

where $f_{\overline{x}} := f(\overline{x}, \cdot)$. One has

$$F^*(u^*, x^*) = (f_{\overline{x}})^*(x^* - u^*) + (a + i_K)^*(u^*), \qquad (u^*, x^*) \in X^* \times X^*,$$

and

$$F^{**}(u, x) = (f_{\overline{x}})^{**}(x) + (a + i_K)^{**}(u + x), \qquad (u, x) \in X \times X.$$

Let us assume that, for every $\overline{x} \in K$, the following conditions hold:

(i) $(\operatorname{dom} f(\overline{x}, \cdot)) \cap (\operatorname{dom} a) \cap K \neq \emptyset$, i.e. $F(0, \cdot)$ is proper;

(ii) $\operatorname{dom}(f_{\overline{x}})^* \neq \emptyset$ and $\operatorname{dom}(a + i_K)^* \neq \emptyset$ or equivalently, $\operatorname{dom} F^* \neq \emptyset$;

(iii) $(f_{\overline{x}})^{**} + (a + i_K)^{**} = (f_{\overline{x}} + a + i_K)^{**}$, i.e. $F^{**}(0, \cdot) = (F(0, \cdot))^{**}$.

Observe that condition (iii) is satisfied in particular when $a \in \Gamma(X)$, $K$ is a closed convex set, and $f(\overline{x}, \cdot) \in \Gamma(X)$ for all $\overline{x} \in K$.

If we apply Theorem 1 to problem $(P)$ we get the following characterization of its solutions:

$\overline{x} \in K$ is a solution of $(P)$ if and only if

$$\left\{ \begin{array}{l} \forall x^* \in \operatorname{dom} b^*, \text{ there exists a net} \\ (u_i^*, x_i^*, \varepsilon_i)_{i\in I} \subset X^* \times X^* \times \mathbb{R} \text{ such that} \\ (f_{\overline{x}})^*(x_i^* - u_i^*) + (a + i_K)^*(u_i^*) + a(\overline{x}) \leq b^*(x^*) + b(\overline{x}) + \varepsilon_i, \ \forall i \in I, \\ \text{and } \lim_{i\in I}(x_i^*, \varepsilon_i) = (x^*, 0_+). \end{array} \right\}$$

Example 3 paves the way to apply Theorem 1 to equilibrium problems, and this will be done in a forthcoming paper.

A striking application of Theorem 1 is the following formula of subdifferential calculus that extends [35, Theorem 2.6.3]. Here $P_{X^*}$ denotes the projection of $U^* \times X^*$ onto $X^*$.

**Theorem 2.** *For any $F : U \times X \to \mathbb{R} \cup \{+\infty\}$ satisfying*

$$F^{**}(0,.) = (F(0,.))^{**} \tag{3.3}$$

*one has*

$$\partial F(0,.)(\overline{x}) = \limsup_{\varepsilon \to 0_+} P_{X^*} \partial_\varepsilon F(0,\overline{x}),\ \forall \overline{x} \in X.$$

*Proof.* We begin with the proof of the inclusion " $\supset$ ". Let $\overline{x} \in X$ and $x^* \in \limsup_{\varepsilon \to 0_+} P_{X^*} \partial_\varepsilon F(0,\overline{x})$. Then, there will exist a net $(u_i^*, x_i^*, \varepsilon_i)_{i \in I} \subset U^* \times X^* \times \mathbb{R}$ such that

$$(u_i^*, x_i^*) \in \partial_{\varepsilon_i} F(0,\overline{x}), \forall i \in I, \text{ and } \lim_{i \in I} (x_i^*, \varepsilon_i) = (x^*, 0_+).$$

We thus have

$$F(u,x) - F(0,\overline{x}) \geq \langle u_i^*, u \rangle + \langle x_i^*, x - \overline{x} \rangle - \varepsilon_i,\ \forall (i,u,x) \in I \times U \times X,$$

and, in particular,

$$F(0,x) - F(0,\overline{x}) \geq \langle x_i^*, x - \overline{x} \rangle - \varepsilon_i,\ \forall (i,x) \in I \times X.$$

Passing to the limit on $i$ for each fixed $x \in X$, we get

$$F(0,x) - F(0,\overline{x}) \geq \langle x^*, x - \overline{x} \rangle,\ \forall x \in X,$$

that is, $x^* \in \partial F(0,.)(\overline{x})$.

We prove now the reverse inclusion " $\subset$ ". Let $\overline{x} \in X$ and $x^* \in \partial F(0,.)(\overline{x})$. This entails $F(0,\overline{x}) \in \mathbb{R}$, $F(0,.)$ is proper, and $\operatorname{dom} F^* \neq \emptyset$. The inclusion now readily follows from Theorem 1 with $h \in \Gamma(X)$ being the affine continuous mapping defined as follows:

$$h(x) := \langle x^*, x - \overline{x} \rangle + F(0,\overline{x}),\ \forall x \in X.$$

Indeed, since $x^* \in \partial F(0,.)(\overline{x})$ we have

$$F(0,.) \geq h,$$

and, by Theorem 1, there exists a net $(u_i^*, x_i^*, \varepsilon_i)_{i \in I} \subset U^* \times X^* \times \mathbb{R}$ such that

$$F^*(u_i^*, x_i^*) \leq \langle x^*, \overline{x} \rangle - F(0,\overline{x}) + \varepsilon_i,\ \forall i \in I,$$

and $(x_i^*, \varepsilon_i) \to (x^*, 0_+)$. According to this,

$$(u_i^*, x_i^*) \in \partial_{\varepsilon_i} F(0,\overline{x}),\ \text{and } (x_i^*, \varepsilon_i) \to (x^*, 0_+),$$

which means

$$x^* \in \limsup_{\varepsilon \to 0_+} P_{X^*} \partial_\varepsilon F(0,\overline{x}).$$

□

From Theorem 2 we obtain the following extension of the Hiriart-Urruty and Phelps formula [13, Corollary 2.1] and of Theorem 13 in [10]. See also [23, Theorem 4] for another approach of this result.

**Proposition 2.** ***(Subdifferential of the sum)*** *Let $f, g : X \to \mathbb{R} \cup \{+\infty\}$ be a couple of functions satisfying*

$$(f+g)^{**} = f^{**} + g^{**}. \tag{3.4}$$

*Then, for any $\overline{x} \in X$,*

$$\partial(f+g)(\overline{x}) = \bigcap_{\varepsilon>0} \operatorname{cl}\left(\partial_\varepsilon f(\overline{x}) + \partial_\varepsilon g(\overline{x})\right).$$

*Proof.* The inclusion " $\supset$ " always holds and it is not difficult to be proved. So, we only have to prove the inclusion " $\subset$ ". Let $\overline{x} \in X$ and $x^* \in \partial(f+g)(\overline{x})$. Setting

$$F(u,x) := f(u+x) + g(x), \ (u,x) \in X^2.$$

We get

$$F(0,.) = f + g. \tag{3.5}$$

Since $\partial(f+g)(\overline{x}) \neq \emptyset$ one has by (3.4)

$$f^{**}(\overline{x}) + g^{**}(\overline{x}) = (f+g)^{**}(\overline{x}) = f(\overline{x}) + g(\overline{x}) \in \mathbb{R}.$$

It follows easily that all the functions $f^*, g^*, f^{**}, g^{**}$ are proper. We have then, straightforwardly,

$$F^*(u^*, x^*) = f^*(u^*) + g^*(x^* - u^*), \ (u^*, x^*) \in (X^*)^2, \tag{3.6}$$

$$F^{**}(u,x) = f^{**}(u+x) + g^{**}(x), \ (u,x) \in X^2, \tag{3.7}$$

and so, by (3.4), (3.5), and (3.7), we have $F^{**}(0,.) = (F(0,.))^{**}$. Since $x^* \in \partial F(0,.)(\overline{x})$, we can thus apply Theorem 2 to conclude the existence of a net $(u_i^*, x_i^*, \varepsilon_i)_{i \in I} \subset (X^*)^2 \times \mathbb{R}$ such that

$$(u_i^*, x_i^*) \in \partial_{\varepsilon_i} F(0, \overline{x}), \text{ and } (x_i^*, \varepsilon_i) \to (x^*, 0_+). \tag{3.8}$$

By (3.6) and (3.8) one has

$$[f^*(u_i^*) + f(\overline{x}) - \langle u_i^*, \overline{x}\rangle] + [g^*(x_i^* - u_i^*) + g(\overline{x}) - \langle x_i^* - u_i^*, \overline{x}\rangle] \leq \varepsilon_i, \ \forall i \in I.$$

Since the the expressions in the two brackets are nonnegative (by Fenchel inequality), each of them is less or equal to $\varepsilon_i$. We thus have $u_i^* \in \partial_{\varepsilon_i} f(\overline{x})$, and $x_i^* - u_i^* \in \partial_{\varepsilon_i} g(\overline{x})$ for all $i \in I$, and so,

$$x^* = \lim_{i \in I}(u_i^* + x_i^* - u_i^*) \in \limsup_{\varepsilon \to 0_+} \left(\partial_\varepsilon f(\overline{x}) + \partial_\varepsilon g(\overline{x})\right) = \bigcap_{\varepsilon>0} \operatorname{cl}\left(\partial_\varepsilon f(\overline{x}) + \partial_\varepsilon g(\overline{x})\right).$$

□

**Remark 1.** *It is worth observing that if $f, g \in \Gamma(X)$, then*

$$(f+g)^{**} = f + g = f^{**} + g^{**}.$$

*Thus, Proposition 2 is a nonconvex version of* [13, Corollary 2.1].

We finish this section with a relevant geometrical characterization of condition (a) in Theorem 1.

**Proposition 3.** *For any $F : U \times X \to \mathbb{R} \cup \{+\infty\}$, the following statements are equivalent*:

(a) $F^{**}(0, \cdot) = (F(0, \cdot))^{**}$ *and it is proper,*
(b) $\emptyset \neq \operatorname{epi}(F(0, \cdot))^* = \operatorname{cl} \bigcup_{u^* \in U^*} \operatorname{epi} F^*(u^*, \cdot) \neq X^* \times \mathbb{R}$.

*Proof.* Let us introduce the marginal dual function

$$\gamma(x^*) = \inf_{u^* \in U^*} F^*(u^*, x^*), \; x^* \in X^*,$$

which is convex [35, Theorem 2.1.3(v)]. Denoting by $\overline{\gamma}$ the $w^*-$lsc hull of $\gamma$, it is well-known that

$$\operatorname{epi} \overline{\gamma} = \operatorname{cl} \bigcup_{u^* \in U^*} \operatorname{epi} F^*(u^*, \cdot), \tag{3.9}$$

and also that [35, Theorem 2.6.1(i)]

$$\gamma^* = F^{**}(0, \cdot). \tag{3.10}$$

Assume that (a) holds. Then, by (3.10) $\gamma^*$ is proper and so $\overline{\gamma} = \gamma^{**}$. Using (3.10) again, we get from (a)

$$\overline{\gamma} = \gamma^{**} = (F(0, \cdot))^{***} = (F(0, \cdot))^*,$$

which yields the properness of $(F(0, \cdot))^*$ and, thanks to (3.9) we obtain (b).

Assume now that (b) holds. By (3.9) we conclude that $\overline{\gamma} = (F(0, \cdot))^*$ and $\overline{\gamma}$ is proper. Since $\overline{\gamma} = \gamma^{**}$, we have $\gamma^{**} = (F(0, \cdot))^*$ and hence, $\gamma^* = \gamma^{***} = (F(0, \cdot))^{**}$. Combining this and (3.10), we get $(F(0, \cdot))^{**} = F^{**}(0, \cdot)$ and the properness of this function as well. □

**Remark 2.** It is worth giving here some observations on the assumptions of Proposition 3.

(i) The statement (a) in Proposition 3 is equivalent to:

(a') $F(0, \cdot)$ is proper, $\operatorname{dom} F^* \neq \emptyset$, and $F^{**}(0, \cdot) = (F(0, \cdot))^{**}$.

(ii) The statement (b) in Proposition 3 holds in particular when $F$ is a proper convex and lsc function such that $0 \in P_U(\operatorname{dom} F)$, where $P_U$ denotes the projection of $U \times X$ onto $U$, since in this case $F^{**}(0, \cdot) = (F(0, \cdot))^{**} = F(0, \cdot)$ and $F(0, \cdot)$ is proper (see [2, Theorem 2]).

## 4. Generalized Farkas lemma for nonconvex systems

This section is addressed to establish necessary and sufficient conditions for asymptotic versions of Farkas lemma for systems without convexity and lower semicontinuity.

Given $H : \operatorname{dom} H \subset X \to U$ and $g : U \to \mathbb{R} \cup \{+\infty\}$, we set

$$(g \circ H)(x) = \begin{cases} g(H(x)), & \text{if } x \in \operatorname{dom} H, \\ +\infty, & \text{if } x \in X \setminus \operatorname{dom} H. \end{cases}$$

We consider a cone $S \subset U$ (i.e., $u \in S$ and $\alpha > 0$ imply $\alpha u \in S$), whose nonnegative polar cone is defined by $S^+$:

$$S^+ := \{u^* \in U^* : \; \langle u^*, u \rangle \geq 0, \; \forall u \in S\}.$$

In contrast with [5], neither lower semicontinuity nor convexity are required for the mapping $u^* \circ H$, with $u^* \in S^+$.

As a consequence of Theorem 1, we get the following versions of the Farkas lemma for nonconvex systems.

**Proposition 4.** *(**Farkas lemma for nonconvex systems I**) Consider $f : X \to \mathbb{R} \cup \{+\infty\}$, $C \subset X$, $H : \operatorname{dom} H \subset X \to U$, and $S$ a cone in $U$. Assume that the two following conditions hold*

$$(\operatorname{dom} f) \cap C \cap H^{-1}(-S) \neq \emptyset, \tag{4.1}$$

$$\begin{aligned} &\exists (u_0^*, x_0^*, \eta_0) \in S^+ \times X^* \times \mathbb{R} \text{ such that} \\ &f(x) + (u_0^* \circ H)(x) \geq \langle x_0^*, x \rangle - \eta_0, \ \forall x \in C. \end{aligned} \tag{4.2}$$

*Then the following statements are equivalent*:

(a) $(f + i_C + i_{-S} \circ H)^{**} = \sup\limits_{u^* \in S^+} (f + i_C + u^* \circ H)^{**}$,

(b) *For any $h \in \Gamma(X)$, we have $(\alpha) \Leftrightarrow (\beta)$ where*

$(\alpha)$ $\quad C \cap H^{-1}(-S) \subset [f - h \geq 0]$,

*and*

$$(\beta) \quad \left\{ \begin{array}{l} \forall x^* \in \operatorname{dom} h^*, \text{ there exists a net} \\ (u_i^*, x_i^*, \varepsilon_i)_{i \in I} \subset S^+ \times X^* \times \mathbb{R} \\ \text{such that} \left\{ \begin{array}{l} (f + i_C + u_i^* \circ H)^*(x_i^*) \leq h^*(x^*) + \varepsilon_i, \ \forall i \in I, \\ \text{and } \lim_{i \in I} (x_i^*, \varepsilon_i) = (x^*, 0_+). \end{array} \right. \end{array} \right.$$

*Proof.* Define $g = f + i_C$ and

$$F(u, x) := g(x) + i_{-S}(H(x) + u), \ (u, x) \in U \times X.$$

(According to our convention, if $x \notin \operatorname{dom} H$, $i_{-S}(H(x) + u) = +\infty$, $\forall u \in U$).

Observe that $F(0, .) = g + i_{-S} \circ H$. Since $S$ is a cone, we get easily

$$F^*(u^*, x^*) = \left\{ \begin{array}{ll} (g + u^* \circ H)^*(x^*), & \text{if } u^* \in S^+, \\ +\infty, & \text{otherwise,} \end{array} \right. \tag{4.3}$$

and so,

$$F^{**}(0, \cdot) = \sup_{u^* \in S^+} (g + u^* \circ H)^{**}.$$

By (4.1) $F(0, .)$ is proper. By (4.2) and (4.3) one has $\operatorname{dom} F^* \neq \emptyset$. Thus the equivalence between (a) and (b) follows directly from Theorem 1. $\square$

Let us now specify a standard situation in which the condition (a) in Proposition 4 is satisfied. To this end one needs the following lemma.

**Lemma 1.** *Assume that the cone $S \subset U$ is closed and convex. Then for any map $H : \operatorname{dom} H \subset X \to U$ one has*

$$i_{-S} \circ H = \sup_{u^* \in S^+} u^* \circ H.$$

*Proof.* We have to prove that for any $x \in \operatorname{dom} H$ one has

$$i_{-S}(H(x)) = \sup_{u^* \in S^+} \langle u^*, H(x) \rangle.$$

If $H(x) \in -S$ then the last equality holds trivially since both sides are equal to zero. If $H(x) \notin -S$, since $S$ is a closed convex cone, the Hahn-Banach theorem yields the existence of $u^* \in S^+$ such that $\langle u^*, H(x) \rangle > 0$. So, $\sup\limits_{n \geq 1} \langle n u^*, H(x) \rangle = +\infty$, and we have

$$i_{-S}(H(x)) = +\infty = \sup_{u^* \in S^+} \langle u^*, H(x) \rangle.$$

$\square$

**Remark 3.** From Lemma 1, it easily follows that the condition (a) in Proposition 4 is in particular satisfied whenever $S$ is a closed convex cone and

$$(f + i_C + u^* \circ H) \in \Gamma(X),\ \forall u^* \in S^+.$$

**Proposition 5** (**Farkas lemma for nonconvex systems II**). *Consider $f : X \to \mathbb{R} \cup \{+\infty\}$, $C \subset X$, $H : \operatorname{dom} H \subset X \to Z$, and $S$ a cone in $Z$. Assume that* (4.1) *holds together with*

$$\begin{aligned}\text{(4.4)}\qquad &\exists\ (u_0^*, y_0^*, t_0^*, x_0^*, \eta_0) \in S^+ \times (X^*)^3 \times \mathbb{R} \text{ such that}\\ &f(y) + (u_0^* \circ H)(x) \geq \langle y_0^*, y\rangle + \langle t_0^*, t\rangle + \langle x_0^* - y_0^* - t_0^*, x\rangle - \eta_0,\\ &\quad \forall (y, t, x) \in X \times C \times \operatorname{dom} H.\end{aligned}$$

*Then the following statements are equivalent*:

(c) $(f + i_C + i_{-S} \circ H)^{**} = f^{**} + i_{\overline{\operatorname{co}} C} + \sup\limits_{u^* \in S^+} (u^* \circ H)^{**}$,

(d) *For any $h \in \Gamma(X)$, one has $(\gamma) \Leftrightarrow (\delta)$ where*

$(\gamma)\quad C \cap H^{-1}(-S) \subset [f - h \geq 0]$,

*and*

$$(\delta) \begin{cases} \forall x^* \in \operatorname{dom} h^*, \text{ there exists a net} \\ (u_i^*, y_i^*, t_i^*, x_i^*, \varepsilon_i)_{i \in I} \subset S^+ \times (X^*)^3 \times \mathbb{R} \text{ such that} \\ \begin{cases} f^*(y_i^*) + i_C^*(t_i^*) + (u_i^* \circ H)^*(x_i^* - y_i^* - t_i^*) \leq h^*(x^*) + \varepsilon_i,\ \forall i \in I, \\ \text{and } \lim_{i \in I}(x_i^*, \varepsilon_i) = (x^*, 0_+). \end{cases} \end{cases}$$

*Proof.* Define now $F : U \times X \to \mathbb{R} \cup \{+\infty\}$ with $U = Z \times X^2$ and

$$F(u, y, t, x) := f(x + y) + i_C(x + t) + i_{-S}(H(x) + u),\ (u, y, t, x) \in U \times X^3.$$

(According to our convention, if $x \notin \operatorname{dom} H$, $F(u, y, t, x) = +\infty$.)

Observe that

$$F(0, 0, 0, \cdot) = f + i_C + i_{-S} \circ H.$$

Since $S$ is a cone, a straightforward computation leads us to

$$\text{(4.5)}\qquad F^*(u^*, y^*, t^*, x^*) = \begin{cases} f^*(y^*) + i_C^*(t^*) + (u^* \circ H)^*(x^* - y^* - t^*), \\ \text{if } (u^*, y^*, t^*, x^*) \in S^+ \times (X^*)^3, \\ +\infty,\ \text{otherwise} \end{cases},$$

and so,

$$F^{**}(0, 0, 0, \cdot) = f^{**} + i_{\overline{\operatorname{co}} C} + \sup_{u^* \in S^+} (u^* \circ H)^{**}.$$

By (4.1), $F(0, 0, 0, \cdot)$ is proper. By (4.4) and (4.5) one has $\operatorname{dom} F^* \neq \emptyset$. Thus the equivalence between (c) and (d) follows directly from Theorem 1. $\square$

**Remark 4.** Propositions 4 and 5 establish necessary and sufficient conditions for Farkas lemma in asymptotic forms and they are new (even for convex data) to the knowledge of the authors. This type of conditions for nonasymptotic form and for the convex, lower semicontinuity systems without set constraint (i.e., where $h \equiv 0$, $C = X$) was proposed recently in [19].

**Corollary 1** ([5, Theorem 3]). *Let $f, h \in \Gamma(X)$, $C$ be a closed convex set in $X$, $S$ a closed convex cone in $Z$, and $H : X \to Z$ a mapping. Assume that* (4.1) *holds together with*

$$\text{(4.6)}\qquad u^* \circ H \in \Gamma(X),\ \forall u^* \in S^+.$$

*Then the following statements $(\gamma)$ and $(\delta)$ in Proposition 5 are again equivalent.*

*Proof.* By Lemma 1 one has

$$i_{-S}\circ H=\sup_{u^*\in S^+}u^*\circ H.$$

By (4.6) we get $i_{-S}\circ H\in\Gamma(X)$ (recall that $H^{-1}(-S)\neq\emptyset$). As $f\in\Gamma(X)$ and $C$ is closed and convex, condition (4.4) holds. To see this, we can simply take $u_0^*=t_0^*=0$, $y_0^*\in\operatorname{dom}f^*$, $x_0^*=y_0^*$, and $\eta_0=f^*(y_0^*)$. It is easy to see that the condition (c) in Proposition 5 holds, too. Consequently, the statement (d) in Proposition 5 is true, and this is precisely what Corollary 1 says. □

**Remark 5.** When $H$ is $S-$convex, i.e. when

$$H(\lambda x+(1-\lambda)y)-\lambda H(x)-(1-\lambda)H(y)\in -S,\ \forall x,y\in X,\forall\lambda\in[0,1],$$

the condition (4.6) is satisfied if $H$ is lower semicontinuous in the following sense (see [27]):

$$\forall x\in X \text{ and } \forall V\in\mathcal{N}(H(x))\ \exists W\in\mathcal{N}(x) \text{ such that } H(W)\subset V+S^+,$$

where $\mathcal{N}(y)$ denotes a neighborhoods basis of $y$.

## 5. Nonconvex optimization problems. Optimality and duality

We consider the nonconvex optimization problem

$$\text{(P)}\qquad \text{minimize } [f(x)-h(x)] \text{ s.t. } x\in C \text{ and } H(x)\in -S,$$

where $f,h:X\to\mathbb{R}\cup\{+\infty\}$, $C\subset X$, $S$ is a cone in $U$, and $H:\operatorname{dom}H\subset X\to U$.

**Proposition 6** (**Optimality condition for** (P))**.** *Consider $f:X\to\mathbb{R}\cup\{+\infty\}$, $C\subset X$, $H:\operatorname{dom}H\subset X\to U$, and $S$ a cone in $U$. Assume that* (4.2) *holds together with*

$$(f+i_C+i_{-S}\circ H)^{**}=\sup_{u^*\in S^+}(f+i_C+u^*\circ H)^{**}. \tag{5.1}$$

*Then for each $h\in\Gamma(X)$ and any $a\in C\cap H^{-1}(-S)\cap\operatorname{dom}f\cap\operatorname{dom}h$, the following statements are equivalent*:

(a) *$a$ is a global optimal solution of* (P)*.*

(b) *$\forall x^*\in\operatorname{dom}h^*$, there exists a net $(u_i^*,x_i^*,\varepsilon_i)_{i\in I}\subset S^+\times X^*\times\mathbb{R}$ such that*

$$(f+i_C+u_i^*\circ H)^*(x_i^*)\le h^*(x^*)+h(a)-f(a)+\varepsilon_i,\ \forall i\in I,$$

*and*

$$\lim_{i\in I}(x_i^*,\varepsilon_i)=(x^*,0_+).$$

*Proof.* This is a straightforward consequence of Proposition 4. Indeed, $a\in C\cap H^{-1}(-S)\cap\operatorname{dom}f\cap\operatorname{dom}h$ is a global optimal solution of (P) if and only if

$$x\in C,\ H(x)\in -S\implies f(x)-[h(x)+f(a)-h(a)]\ge 0,$$

and this happens if and only if the statement ($\alpha$) in Proposition 4 holds with $\tilde{h}$, defined as $\tilde{h}(x):=h(x)+f(a)-h(a)$, instead of $h$. The conclusion follows from Proposition 4, taking into account the fact that $\tilde{h}^*(x^*)=h^*(x^*)-f(a)+h(a)$. □

The following optimality condition is a consequence of Proposition 5. The proof follows the same line as that of Proposition 6 and, therefore, it will be omitted.

**Proposition 7** (**Optimality condition for** (P))**.** *Consider $f : X \to \mathbb{R} \cup \{+\infty\}$, $C \subset X$, $S$ a cone in $U$, and $H : \operatorname{dom} H \subset X \to U$. Assume that* (4.4) *holds together with*

$$(f + i_C + i_{-S} \circ H)^{**} = f^{**} + i_{\overline{\text{co}}C} + \sup_{u^* \in S^+} (u^* \circ H)^{**}. \tag{5.2}$$

*Then for each $h \in \Gamma(X)$ and $a \in C \cap H^{-1}(-S) \cap \operatorname{dom} f \cap \operatorname{dom} h$, the following statements are equivalent*:

(a) *$a$ is a global optimal solution of* (P),

(b) *$\forall x^* \in \operatorname{dom} h^*$, there exists a net $(u_i^*, y_i^*, t_i^*, x_i^*, \varepsilon_i)_{i \in I} \subset S^+ \times (X^*)^3 \times \mathbb{R}$ such that*

$$f^*(y_i^*) + i_C^*(t_i^*) + (u_i^* \circ H)^*(x_i^* - y_i^* - t_i^*) \leq h^*(x^*) + h(a) - f(a) + \varepsilon_i, \ \forall i \in I,$$

*and*

$$\lim_{i \in I} (x_i^*, \varepsilon_i) = (x^*, 0_+).$$

**Corollary 2** ([5, Proposition 2])**.** *Let $f, h \in \Gamma(X)$, $C$ be a closed convex set in $X$, $S$ a closed convex cone in $U$, and $H : X \to U$ a mapping. Assume additionally that (4.6) holds. Then, for each $a \in C \cap H^{-1}(-S) \cap \operatorname{dom} f \cap \operatorname{dom} h$, the statements (a) and (b) in Proposition 7 are equivalent.*

*Proof.* By Lemma 1 and (4.6) one has

$$i_{-S} \circ H = \sup_{u^* \in S^+} u^* \circ H \in \Gamma(X),$$

(recall that $H^{-1}(-S) \neq \emptyset$ as $a \in H^{-1}(-S)$). Since $f \in \Gamma(X)$ and $C$ is closed and convex, conditions (4.4) and (5.2) in Proposition 7 hold (see the proof of Corollary 1). Therefore, statements (a) and (b) in Proposition 7 are equivalent. □

**Proposition 8** (**Duality theorem for** (P))**.** *Let $f : X \to \mathbb{R} \cup \{+\infty\}$, $h \in \Gamma(X)$, $C \subset X$, $S \subset U$, and $H : \operatorname{dom} H \subset X \to U$ be as in Proposition 6 (i.e. satisfying* (4.2) *and* (5.1)*). Moreover, assume that $\alpha := \inf(\mathrm{P}) \in \mathbb{R}$. Then it holds*:

$$\inf(\mathrm{P}) = \inf_{x^* \in \operatorname{dom} h^*} \sup_{\substack{(u_i^*)_{i \in I} \subset S^+ \\ (x_i^*)_{i \in I} \subset X^* \\ x_i^* \to x^*}} \left[ h^*(x^*) - \limsup_{i \in I} (f + i_C + u_i^* \circ H)^*(x_i^*) \right]. \tag{5.3}$$

*Proof.* We begin with the inequality [≤]. Take $x^* \in \operatorname{dom} h^*$ and observe that

$$x \in C, \ H(x) \in -S \ \Rightarrow \ f(x) - [h(x) + \alpha] \geq 0.$$

By Proposition 4, with $\tilde{h}(x) := h(x) + \alpha$ playing the role of $h$, the previous inequality implies the existence of a net $(u_i^*, x_i^*, \varepsilon_i)_{i \in I} \subset S^+ \times X^* \times \mathbb{R}$ such that

$$(f + i_C + u_i^* \circ H)^*(x_i^*) \leq h^*(x^*) - \alpha + \varepsilon_i, \ \forall i \in I,$$

and

$$\lim_{i \in I} (x_i^*, \varepsilon_i) = (x^*, 0_+),$$

which in fact entails

$$\limsup_{i \in I} (f + i_C + u_i^* \circ H)^*(x_i^*) \leq h^*(x^*) - \alpha,$$

and thus,

$$\inf(\mathrm{P}) \leq \sup_{\substack{(u_i^*, x_i^*)_{i\in I} \subset S^+ \times X^* \\ x_i^* \to x^*}} \left\{ h^*(x^*) - \limsup_{i\in I} (f + i_C + u_i^* \circ H)^*(x_i^*) \right\},$$

for all $x^* \in \operatorname{dom} h^*$, so the inequality $[\leq]$ in (5.3) holds.

We now prove the inequality $[\geq]$ in (5.3). If $x^* \in \operatorname{dom} h^*$, for any net $(u_i^*, x_i^*)_{i\in I} \subset S^+ \times X^*$ such that $x_i^* \to x^*$, one has

$$(f + i_C + u_i^* \circ H)^*(x_i^*) \geq \langle x_i^*, x\rangle - f(x) - \langle u_i^*, H(x)\rangle, \ \forall i \in I, \forall x \in C \cap \operatorname{dom} H,$$

and since $(u_i^*)_{i\in I} \subset S^+$,

$$(f + i_C + u_i^* \circ H)^*(x_i^*) \geq \langle x_i^*, x\rangle - f(x), \ \forall i \in I, \forall x \in C \cap H^{-1}(-S).$$

It follows then that, $\forall i \in I, \ \forall x \in C \cap H^{-1}(-S)$,

$$h^*(x^*) - \limsup_{i\in I} (f + i_C + u_i^* \circ H)^*(x_i^*) \leq h^*(x^*) - \langle x^*, x\rangle + f(x),$$

and so,

$$\begin{aligned} &\sup_{\substack{(u_i^*, x_i^*)_{i\in I} \subset S^+ \times X^* \\ x_i^* \to x^*}} \left\{ h^*(x^*) - \limsup_{i\in I} (f + i_C + u_i^* \circ H)^*(x_i^*) \right\} \\ \leq \quad & h^*(x^*) - \langle x^*, x\rangle + f(x), \ \forall i \in I, \forall x \in C \cap H^{-1}(-S). \end{aligned}$$

Now, since $x^*$ is an arbitrary element of $\operatorname{dom} h^*$, we get by taking the infimum on $x^* \in \operatorname{dom} h^*$ in the last inequality, that the right hand side of (5.3) is less or equal to

$$f(x) - h^{**}(x) = f(x) - h(x), \ \forall x \in C \cap H^{-1}(-S),$$

so that finally the inequality $[\geq]$ in (5.3) holds. □

Now we derive from (5.3) another duality formula for (P) in which we denote by

$$L(u^*, x) := f(x) + (u^* \circ H)(x), \ (u^*, x) \in S^+ \times X$$

the *Lagrange function* associated with $f$ and $H$.

**Corollary 3.** *With the same assumptions as in Proposition 8, one also has*

$$\inf(\mathrm{P}) = \inf_{x^* \in \operatorname{dom} h^*} \sup_{(u_i^*)_{i\in I} \subset S^+} \inf_{x\in C} \left\{ h^*(x^*) - \langle x^*, x\rangle + \liminf_{i\in I} L(u_i^*, x) \right\}.$$

*Proof.* By (5.3) one easily gets

$$\inf(\mathrm{P}) \leq \inf_{x^* \in \operatorname{dom} h^*} \sup_{\substack{(u_i^*, x_i^*)_{i\in I} \subset S^+ \times X^* \\ x_i^* \to x^*}} \inf_{x\in C} \left\{ h^*(x^*) + \liminf_{i\in I} (L(u_i^*, x) - \langle x_i^*, x\rangle) \right\}.$$

Since $x_i^* \to x^*$, one has

$$\liminf_{i\in I} (L(u_i^*, x) - \langle x_i^*, x\rangle) = \left( \liminf_{i\in I} L(u_i^*, x) \right) - \langle x^*, x\rangle,$$

and so,

$$\inf(\mathrm{P}) \leq \inf_{x^* \in \operatorname{dom} h^*} \sup_{(u_i^*)_{i\in I} \subset S^+} \inf_{x\in C} \left\{ h^*(x^*) - \langle x_i^*, x\rangle + \liminf_{i\in I} L(u_i^*, x) \right\} =: \beta.$$

In order to prove the opposite inequality, we have to check that for every $\overline{x} \in C \cap H^{-1}(-S)$

$$\begin{aligned} f(\overline{x}) - h(\overline{x}) &= f(\overline{x}) - h^{**}(\overline{x}) \\ &= \inf_{x^* \in \operatorname{dom} h^*} \{f(\overline{x}) + h^*(x^*) - \langle x^*, \overline{x} \rangle\} \\ &\geq \beta, \end{aligned}$$

and this happens if, for every $\overline{x} \in C \cap H^{-1}(-S)$ and every $\overline{x}^* \in \operatorname{dom} h^*$, we have

$$f(\overline{x}) + h^*(\overline{x}^*) - \langle \overline{x}^*, \overline{x} \rangle \geq \beta.$$

In fact we have

$$\begin{aligned} \beta &\leq \sup_{(u_i^*)_{i\in I} \subset S^+} \inf_{x \in C} \left\{ h^*(\overline{x}^*) - \langle \overline{x}^*, x \rangle + \liminf_{i \in I} L(u_i^*, x) \right\} \\ &\leq \sup_{(u_i^*)_{i\in I} \subset S^+} \left\{ h^*(\overline{x}^*) - \langle \overline{x}^*, \overline{x} \rangle + \liminf_{i \in I} L(u_i^*, \overline{x}) \right\}, \end{aligned}$$

and since $(u_i^*, \overline{x})_{i \in I} \subset S^+ \times H^{-1}(-S)$, one has

$$L(u_i^*, \overline{x}) = f(\overline{x}) + (u_i^* \circ H)(\overline{x}) \leq f(\overline{x}),$$

so that we are done. □

**Corollary 4** ([5, Proposition 7], [6])**.** *Assume that $f \in \Gamma(X)$, $C$ is a closed convex set in $X$, $S$ a closed convex cone in $U$, $H : X \to U$ satisfies* (4.6)*, and* $(\operatorname{dom} f) \cap C \cap H^{-1}(-S) \neq \emptyset$. *Then*

$$\begin{aligned} \inf_{x \in C \cap H^{-1}(-S)} f(x) &= \sup_{(u_i^*)_{i\in I} \subset S^+} \inf_{x \in C} \liminf_{i \in I} L(u_i^*, x) \\ &= \inf_{x \in C} \sup_{(u_i^*)_{i\in I} \subset S^+} \liminf_{i \in I} L(u_i^*, x). \end{aligned}$$

*Proof.* Since $L(u_i^*, x) := f(x) + (u_i^* \circ H)(x) \leq f(x)$, for any $(u_i^*, x)_{i \in I} \subset S^+ \times H^{-1}(-S)$, it is easy to see that

$$\inf_{x \in C} \sup_{(u_i^*)_{i\in I} \subset S^+} \liminf_{i \in I} L(u_i^*, x) \leq \inf_{x \in C \cap H^{-1}(-S)} f(x).$$

Observe also that

$$\alpha := \inf_{x \in C \cap H^{-1}(-S)} f(x) \leq \sup_{(u_i^*)_{i\in I} \subset S^+} \inf_{x \in C} \liminf_{i \in I} L(u_i^*, x).$$

This is obvious if $\alpha = -\infty$. Note that the assumptions of the corollary imply that (4.2) and (5.1) hold and so, if $\alpha \in \mathbb{R}$, the last inequality comes from Corollary 3 (applied with $h = 0$), and from the fact that that $\alpha < +\infty$.

On the other hand, since

$$\sup_{(u_i^*)_{i\in I} \subset S^+} \inf_{x \in C} \liminf_{i \in I} L(u_i^*, x) \leq \inf_{x \in C} \sup_{(u_i^*)_{i\in I} \subset S^+} \liminf_{i \in I} L(u_i^*, x),$$

we are done. □

By taking $H = 0$ in (P) we get the problem

$$(\mathrm{P}_1) \qquad \text{minimize } [f(x) - h(x)] \text{ s.t. } x \in C.$$

So, it is not surprising that the previous results cover, as a special case, the well-known duality for DC problems [31] (see, also, [28] and [32]). For instance, from

Corollary 3 with $H = 0$ and $C = X$ we straightforwardly get that, for any $h \in \Gamma(X)$ and any $f : X \to \mathbb{R} \cup \{+\infty\}$ with $f^*$ proper, one has

$$\inf_{x \in X} \{f(x) - h(x)\} = \inf_{x^* \in X^*} \{h^*(x^*) - f^*(x^*)\}, \tag{5.4}$$

which still holds when $f^*$ is not proper.

According to Proposition 7 we provide next a characterization of the optimal solution set for the problem $(\mathrm{P}_1)$.

**Proposition 9.** *Let $h \in \Gamma(X)$, $C \subset X$, and $f : X \to \mathbb{R} \cup \{+\infty\}$ be such that $f^*$ proper and*

$$(f + i_C)^{**} = f^{**} + i_{\overline{\mathrm{co}}C}. \tag{5.5}$$

*Then, for any $a \in C \cap \operatorname{dom} f \cap \operatorname{dom} h$, the following statements are equivalent*:

(a) *$a$ is a global minimum of* $(\mathrm{P}_1)$,

(b) *$\forall x^* \in \operatorname{dom} h^*$, there exists a net $(x_i^*, y_i^*, \varepsilon_i)_{i \in I} \subset (X^*)^2 \times \mathbb{R}$ such that*

$$f^*(y_i^*) + i_C^*(x_i^* - y_i^*) + f(a) \le h^*(x^*) + h(a) + \varepsilon_i, \ \forall i \in I,$$

*and*

$$(x_i^*, \varepsilon_i) \to (x^*, 0_+).$$

*Proof.* It follows from Proposition 7, by taking $H \equiv 0$. □

**Remark 6.** Condition (5.5) is in particular satisfied in the following two important cases:

(i) $f \in \Gamma(X)$, $C$ is closed and convex,

(ii) $C = X$.

Relatively to the case (ii) above we have:

**Proposition 10.** *Let $h \in \Gamma(X)$ and $f : X \to \mathbb{R} \cup \{+\infty\}$ with $f^*$ is proper. Then, for any $a \in C \cap \operatorname{dom} f \cap \operatorname{dom} h$, the following statements are equivalent:*

(a) *$a$ is a global minimum of $f - h$ on $X$,*

(b) *$\forall x^* \in \operatorname{dom} h^*$,*

$$f^*(x^*) + f(a) \le h^*(x^*) + h(a),$$

(c) *$\forall x^* \in \operatorname{dom} h^*$, there exists a net $(x_i^*, \varepsilon_i)_{i \in I} \subset X^* \times \mathbb{R}$ such that*

$$f^*(x_i^*) + f(a) \le h^*(x^*) + h(a) + \varepsilon_i, \ \forall i \in I,$$

*and*

$$(x_i^*, \varepsilon_i) \to (x^*, 0_+).$$

*Proof.* $[(a) \Rightarrow (b)]$ Let $x^* \in \operatorname{dom} h^*$. For any $x \in X$, it holds

$$h^*(x^*) + h(a) \ge \langle x^*, x\rangle - h(x) + h(a) \ge \langle x^*, x\rangle - f(x) + f(a)$$

and we get (b) by taking the supremum over $x \in X$.

$[(b) \Rightarrow (c)]$ Take $x_i^* = x^*$, $\varepsilon_i = 0$, for all $i \in I$ (an arbitrary directed set).

$[(c) \Rightarrow (a)]$ Apply Proposition 9 with $C = X$. □

**Remark 7.** The equivalence of (a) and (b) also follows from (5.4).

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